Lecture Notes on the Character Table of $\mathbf{A_5}$

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Contents

1 Introduction and Overview

The alternating group A_5 is the group of all even permutations on 5 elements. It has order 60 and is famously the smallest nonabelian simple group. These notes will culminate in presenting the *character table* of A_5. The character table is a powerful device in group theory and representation theory, allowing one to:

- Understand all possible irreducible representations (irreps) over $\mathbb{C}$.
- Quickly see how each conjugacy class is represented under each irrep.

- Use orthogonality relations to derive, check, and interpret the character values.

Along the way, we will see:

- Why A_5 has exactly 5 conjugacy classes (hence 5 irreps).
- How the dimensions of these irreps must be $1, 3, 3, 4, 5$.
- Why the character values on the 5-cycles often involve the number $\frac{-1\pm\sqrt{5}}{2}$.

These notes are designed to be as self-contained as possible. For further reading or more detailed proofs, see the references at the end.

2 Review of Group-Theoretic Background

2.1 Conjugacy Classes

A *group* G is a set equipped with a binary operation (written multiplicatively) satisfying closure, associativity, identity, and inverses. Two elements $g, h \in G$ are *conjugate* if there exists $x \in G$ such that $h = xgx^{-1}$. The set of all elements conjugate to g forms the *conjugacy class* of g, denoted $\mathrm{Cl}(g)$. Elements in the same conjugacy class share many important properties (e.g., order).

2.2 The Group A_5 and Its Conjugacy Classes

- **Definition of A_5.** A_5 is the subgroup of the symmetric group S_5 consisting of all even permutations of 5 elements.
- **Order.** $|A_5| = 60$.
- **Simplicity.** A_5 is the smallest nonabelian simple group. This implies it has no non-trivial normal subgroups.
- **Conjugacy classes in A_5.** From direct counting or standard theorems on permutation groups, A_5 has exactly 5 conjugacy classes. Concretely:

  1. The identity permutation (class size 1).
  2. Double transpositions $(ab)(cd)$ (class size 15).
  3. 3-cycles (abc) (class size 20).
  4. One class of 5-cycles $(abcde)$ (class size 12).
  5. Another class of 5-cycles (also size 12).

  The reason there are two classes of 5-cycles in A_5 (rather than just one in S_5) is that a 5-cycle and its inverse are in the same conjugacy class in S_5, but they split in A_5 because one is an even permutation, the other is also even, but they are not conjugate by an *even* permutation alone.

3 Representations and Characters

3.1 Definitions

Definition 3.1. A (complex) *representation* of a finite group G is a group homomorphism

$$\rho : G \;\to\; \mathrm{GL}(V),$$

where V is a finite-dimensional vector space over $\mathbb{C}$. Equivalently, V becomes a $\mathbb{C}$-vector space on which G acts linearly.

Definition 3.2. The *character* of a representation ρ is the function $\chi : G \to \mathbb{C}$ given by

$$\chi(g) \;=\; \mathrm{Trace}\big(\rho(g)\big).$$

Characters are *class functions*, meaning $\chi(xgx^{-1}) = \chi(g)$. Hence, character values are constant on each conjugacy class.

Definition 3.3. A representation is *irreducible* (or an *irrep*) if it has no nontrivial, proper G-invariant subspaces. Over $\mathbb{C}$, every finite-dimensional representation breaks into a direct sum of irreps (Maschke's Theorem).

3.2 Irreducible Representations

- **Key Fact:** The number of irreps of a finite group G over $\mathbb{C}$ equals the number of conjugacy classes of G. So since A_5 has 5 conjugacy classes, it has exactly 5 irreps.
- **Dimensions:** Let the irreps be $\rho_1, \rho_2, \ldots, \rho_5$, with dimensions $d_1, \ldots, d_5$. One of them is always the *trivial representation* (dimension 1). By a theorem in character theory (often called the "sum of squares" theorem), we have

$$d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2 = |G| = 60.$$

3.3 Orthogonality Relations

For irreps χ_i, χ_j of a finite group G, the *orthogonality of characters* states:

$$\frac{1}{|G|} \sum_{g \in G} \chi_i(g)\,\overline{\chi_j(g)} \;=\; \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Because characters are class functions, this can be written in terms of sums over conjugacy classes:

$$\frac{1}{|G|} \sum_{k=1}^{r} |C_k|\; \chi_i(C_k)\,\overline{\chi_j(C_k)} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

These orthogonality relations are the primary algebraic tool for determining unknown character values once the dimensions and class sizes are known.

4 Constructing the Character Table of A_5

4.1 Conjugacy Classes of A_5

Recall the 5 classes and their sizes:

1. **Identity** (e) – size 1.
2. **Double Transpositions** $(ab)(cd)$ – size 15.
3. **3-cycles** (abc) – size 20.
4. **5-cycles** (class 5_1) – size 12.
5. **5-cycles** (class 5_2) – size 12.

4.2 Dimensions of Irreps and the Sum of Squares Theorem

We have 5 irreps. Let their dimensions be d_1, d_2, d_3, d_4, d_5. Then:

$$d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2 = 60.$$

- One representation is the trivial representation, so $d_1 = 1$.
- Because A_5 is simple, it has no nontrivial 1-dimensional representations other than the trivial. Hence, $d_1 = 1$ is unique.
- By known classification or by systematically trying integer solutions, the set of dimensions is $\{1, 3, 3, 4, 5\}$. (This can also be found by looking at known group actions, e.g., the action on the cosets of a subgroup, or on geometric objects such as the icosahedron.)

Hence we label the irreps so that:

$$\chi_1(e) = 1, \quad \chi_2(e) = 3, \quad \chi_3(e) = 3, \quad \chi_4(e) = 4, \quad \chi_5(e) = 5.$$

4.3 Orthogonality Constraints and Solving for Character Values

Let us denote the character values on each conjugacy class C_k by $\chi_i(C_k)$. We already know:

- $\chi_1(C_k) = 1$ for all k, since it is the trivial character.
- $\chi_i(e) = d_i$.

Next, we use:

1. The sum of character values (weighted by class sizes) must be zero for each nontrivial irrep if you sum over the entire group (because $\sum_{g \in G} \chi_i(g) = 0$ for $i \neq 1$, a direct consequence of orthogonality with the trivial character).

2. The column-orthogonality condition: for any $i \neq j$,

$$\sum_{k=1}^{5} |C_k| \, \chi_i(C_k) \, \overline{\chi_j(C_k)} = 0.$$

3. Reality conditions or known geometric constructions can also reduce the guesswork. For instance, certain irreps can be realized by permutation representations on 5 or 6 points, or via the action on vertices of a Platonic solid, giving partial or complete information about the trace (character) values.

A systematic (but somewhat tedious) approach is to set up a system of linear equations in the unknown character values and solve. One ultimately finds a table (up to reordering of classes and irreps) like:

	e	$(ab)(cd)$	(abc)	5_1	5_2
χ_1	1	1	1	1	1
χ_2	3	1	0	α_1	α_2
χ_3	3	1	0	α_2	α_1
χ_4	4	0	1	-1	-1
χ_5	5	-1	-1	0	0

where

$$\alpha_1 = \frac{-1+\sqrt{5}}{2}, \quad \alpha_2 = \frac{-1-\sqrt{5}}{2}.$$

Alternatively, some references write α_1 and α_2 in terms of 5th roots of unity:

$$\alpha_1 = 1 + \zeta + \zeta^4, \quad \alpha_2 = 1 + \zeta^2 + \zeta^3,$$

where $\zeta = e^{2\pi i/5}$. Indeed, it is a standard identity that $1 + \zeta + \zeta^4 = \frac{-1+\sqrt{5}}{2}$, etc.
Note: The two 3-dimensional irreps are "conjugate" to each other in a sense, so their character values on the 5-cycle classes are conjugate expressions ($\alpha_1 \leftrightarrow \alpha_2$).

5 Why $\sqrt{5}$?

The appearance of $\sqrt{5}$ in the 3-dimensional representations can be understood by looking at the *eigenvalues* of a 5-cycle under these representations. A 5-cycle in a suitable representation might act as a matrix whose eigenvalues are among the 5th roots of unity. The trace of such a matrix is the sum of some subset of these roots of unity, which often simplifies to expressions involving the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$.

Concretely, if $\zeta = e^{2\pi i/5}$, then

$$\zeta + \zeta^4 = -1 + \frac{\sqrt{5}}{2}, \quad \zeta^2 + \zeta^3 = -1 - \frac{\sqrt{5}}{2}.$$

Such expressions arise naturally in the character values on the 5-cycle classes.

6 Summary and Further Examples

6.1 Summary

- **Conjugacy Classes:** A_5 has 5 classes: identity, double transpositions, 3-cycles, and two classes of 5-cycles.
- **Irreps:** There are 5 irreps (one per conjugacy class), with dimensions $1, 3, 3, 4, 5$.
- **Character Table:** By applying orthogonality relations and known dimensions, we fill out the table. Some entries on the 5-cycle classes involve $\sqrt{5}$.

6.2 Further Examples and Exercises

1. **Permutation Representations:** Show that the permutation representation of A_5 on 5 points decomposes into a direct sum of the trivial representation plus a 4-dimensional representation. Identify the character of that 4-dimensional component.
2. **Action on the Icosahedron:** A_5 is the rotational symmetry group of the icosahedron. Explore how the 3D, 4D, and 5D representations might arise from this viewpoint (e.g., the space of harmonic polynomials on the sphere restricted to icosahedral symmetry).
3. **Comparison to S_5:** Write down the character table of S_5 (which has 7 conjugacy classes) and see how the 5-cycle class does not split in S_5. Compare with the splitting in A_5.

7 References

- **Serre, J.-P.** *Linear Representations of Finite Groups.* Springer.
- **Fulton, W. and Harris, J.** *Representation Theory: A First Course.* Springer.
- **Dummit, D. and Foote, R.** *Abstract Algebra.* Wiley.
- **Lang, S.** *Algebra.* Addison-Wesley.